A Material Point Method for Snow Simulation

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Introduction

- This handles difficult snow behaviors in a **single solver**.

- Contributions
  - We develop a semi-implicit Material Point Method (MPM)
    - Designed to efficiently treat the wide range
  - This is the first time MPM has been used in graphics
Related work

- Geometric snow modeling
  - The bulk of graphics snow is devoted to modeling accumulation

- Granular materials
  - Traditionally, snow in graphics is thought of as a granular material

- Engineering modeling of snow & Elasto-plastic continuum
  - Elasto-plastic constitutive relation worked well
Paper Overview
- Why do they choose MPM?

• Snow dynamics modeling is difficult
  ▪ Due to snow’s variability, usually stemming from environmental
  ▪ Our model ignores root causes

• Strength of MPM
  ▪ Relying on the continuum approximation
  ▪ Avoiding the need to model every snow grain
  ▪ Making self-collisions automatic
  ▪ Better able to handle the dynamics of snow
### Paper Overview
- Comparison with Other Methods

<table>
<thead>
<tr>
<th>Method</th>
<th>Volume Preservation</th>
<th>Stiffness</th>
<th>Plasticity</th>
<th>Fracture</th>
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<td>Reeve particles</td>
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<td>Rigid bodies</td>
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<td>Mesh-based solids</td>
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- In volume preservation, Grid is more effective than MPM
  - Snow is compressible

- In stiffness, FEM is more effective than MPM
  - Remeshing is required with extreme deformation

- MPM is almost ideal for plasticity
  - The inaccuracies of the deformation gradient accumulate as **artificial plasticity**
Paper Overview
- What is Deformation gradient?

- Deformation Gradient $F = \frac{\partial x(X)}{\partial X}$
  - $X$ is the undeformed reference configuration
  - $x(X)$ is the deformed current configuration

- Examples

Reference
http://www.continuummechanics.org
Material point method
- Governing equation

- Mass Preservation
  \[ \frac{D\rho}{Dt} = 0 \]

- Momentum Preservation
  \[ \rho \frac{Dv}{Dt} = \nabla \cdot \sigma + \rho g \]
  - Cauchy stress \( \sigma = \frac{1}{J} \frac{\partial \Psi}{\partial F_E} F_E^T \)
    - \( \Psi \) is elasto-plastic potential energy density
    - \( J = \det(F) \) & \( F = F_E F_P \)
  - If material is isotropic Newtonian fluid \( \sigma_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \)
Material point method
- Particles (Lagrangian)

- Particle (material point) \( p \) holds
  - Position \( x_p \)
  - Velocity \( v_p \)
  - Mass \( m_p \)
  - Deformation gradient \( F_p \)

- The Lagrangian treatment simplifies the discretization of
  - \( \frac{D}{Dt} \rho \)
  - \( \rho \frac{Dv}{Dt} \)
Material point method  
- Background grid (Eulerian)

- Background grid node $i$ holds
  - Constant Position $\mathbf{x}_i$
  - Velocity $\mathbf{v}_i$
  - Mass $m_i$

- Grid is needed for the computation of stress-based force
  - $\nabla \cdot \sigma$ terms in the standard FEM
  - Using cubic B-splines kernel

$$N(x) = \begin{cases} 
\frac{1}{2} |x|^3 - x^2 + \frac{2}{3}, & 0 \leq |x| < 1 \\
-\frac{1}{6} |x|^3 + x^2 - 2|x| + \frac{4}{3}, & 1 \leq |x| < 2 \\
0, & \text{otherwise}
\end{cases}$$

- $w_{ip}(\mathbf{x}_i, \mathbf{x}_p, h) = N(\frac{1}{h} (x_p - x_i)) N(\frac{1}{h} (y_p - y_i)) N(\frac{1}{h} (z_p - z_i))$
  - $h$ is Kernel radius
Material point method
- Full method overview

Particle Domain (Lagrangian)

1. Grid velocities and mass
2. Particle states
3. Particle volumes
4. Grid forces
5. Grid velocities (RHS)
6. Collided grid velocities
7. Updated deformation gradients
8. Particle velocities
9. Collided particles
10. Updated positions

Material Point Method Overview

GridView (Eulerian)
Material point method
- Full method overview

• Rasterize particle data to the grid
  ▪ Mass \( m_i^n = \sum_p m_p w_{ip}^n \)
  ▪ Velocity \( v_i^n = \sum_p v_p^m m_p w_{ip}^n / m_i^n \)

• Compute particle volumes and densities
  ▪ Only once

• Compute grid forces
  ▪ Stress-based forces

• Update velocities on grid
• Grid-based body collisions
• Solve the linear system
• Update deformation gradient
• Update particle velocities and positions
Material point method
- Method flow comparison with FLIP

• Rasterize particle data to the grid
  ▪ Make velocity field

• Compute grid forces
  ▪ MPM: Stress-based forces
  ▪ FLIP: Pressure forces

• Update velocity field
  ▪ \( v^{n+1} = v^n + \Delta t m^{-1} f \)

• Update particle velocity
Constitutive model
- Elasto–plastic energy density function

- Energy density function is

$$\Psi(F_E, F_P) = \mu(F_P)||F_E - R_E||_F^2 + \frac{\lambda(F_P)}{2}(J_E - 1)^2$$

- \( F_E = R_ES_E \) by the polar decomposition
- 1\textsuperscript{st} term of the right side : Shape based
- 2\textsuperscript{nd} term of the right side : Volume based

- Functions of the \( F_P \)
  - \( \mu(F_P) = \mu_0 e^{\xi(1-J_P)} \)
  - \( \lambda(F_P) = \lambda_0 e^{\xi(1-J_P)} \)
- \( \mu_0 \) and \( \lambda_0 \) are initial parameters
- \( \xi \) is plastic hardening parameter
Stress-based forces
- Forces derivation from Total potential energy

- The total elastic potential energy can be expressed as
  \[ \Phi = \int_{\Omega^0} \Psi(F_E(X), F_P(X)) \, dX = \sum_p V_p^0 \psi_p(F_{Ep}, F_{Pp}) \]
  - \(\Omega^0\) is the reference configuration

- Spatial discretization of the stress-based forces
  \[ -f_i = \frac{\partial \Phi(x)}{\partial x_i} = \sum_p V_p^0 \frac{\partial \psi_p}{\partial F_{Ep}} \left( F_{Ep} \right)^T \nabla w_{ip} \]

- Expression in terms of the Cauchy stress and sequence index
  \[ f_i^n = -\sum_p V_p^n \sigma_p \nabla w_{ip} \]
  - Cauchy stress \(\sigma_p = \frac{1}{J_p^n} \frac{\partial \psi_p}{\partial F_{Ep}} \left( F_{Ep} \right)^T \)
  - Occupied volume \(V_p^n = J_p^n V_p^0\)
Stress-based forces
- Predicted quantities

- Predicted grid node position
  - $\hat{x}_i(v_i) = x_i + \Delta t v_i$

- Predicted Deformation Gradient
  - $\hat{F}_{Ep}^{n+1}(\hat{x}) = \left(I + \sum_i (\hat{x}_i - x_i)(\nabla w_{ip}^n)^T\right)F_{Ep}^n$

- Predicted grid forces
  - $f_i^n = f_i(\hat{x}_i(0))$
  - $f_i^{n+1} = f_i(\hat{x}_i(v^{n+1}))$
Stress-based forces
- Semi-implicit update(1)

- Grid velocity Update

\[ v_{i}^{n+1} = v_{i}^{n} + \Delta t m_{i}^{-1} ((1 - \beta) f_{i}^{n} + \beta f_{i}^{n+1}) \]

- \( \beta \) is choose between
  - 0 : Explicit
  - 0~1: Semi implicit
  - 1 : Backward Euler

\[ v_{i}^{n+1} \approx v_{i}^{n} + \Delta t m_{i}^{-1} (f_{i}^{n} + \beta \Delta t \Sigma_{j} \frac{\partial f_{j}^{n}}{\partial \hat{x}_{j}} v_{j}^{n+1}) \]

- \( f_{i}^{n+1} = f_{i}(\hat{x}_{i}(v^{n+1})) = f_{i}(\hat{x}_{i}(0)) + \delta f_{i}(v^{n+1}) \)

- \( \delta f_{i}(v^{n+1}) \approx \Delta t \Sigma_{j} \frac{\partial f_{j}^{n}}{\partial \hat{x}_{j}} v_{j}^{n+1} \)
Stress-based forces
- Semi-implicit update(2)

• The result of the previous page leads to linear system
  \[ \Sigma_j \left( I \delta_{ij} + \beta \Delta t^2 m_i^{-1} \frac{\partial^2 \Phi^n}{\partial \hat{x}_i \partial \hat{x}_j} \right) v_j^{n+1} = v_i^* \]

\[ A \cdot x = b \]

• \[ v_i^* = v_i^n + \Delta t m_i^{-1} f_i^n \]
• Using the conjugate residual method
Deformation gradient update

• Step 1. From page 15(PPT 26) Temporarily defining
  \[ \hat{F}_{Ep}^{n+1} = (I + \Delta t \nabla v_p^{n+1}) F_{Ep}^n, \quad \hat{F}_{pp}^{n+1} = F_{pp}^n \]
  \[ F_p^{n+1} = \hat{F}_{Ep}^{n+1} \hat{F}_{pp}^{n+1} \]
  • All the changes get attributed to the elastic part

• Step 2. Clamping singular values of \( \hat{F}_{Ep}^{n+1} \) to the permitted range
  \[ \hat{F}_{Ep}^{n+1} = U_p \hat{\Sigma}_p V_p^T, \text{ then } \Sigma_p = \text{clamp}(\hat{\Sigma}_p, [1 - \theta_c, 1 + \theta_s]) \]
  • \( \theta_c \) : Critical compression
  • \( \theta_s \) : Critical stretch

• Step 3. Push changes of the elastic part into the plastic part
  \[ F_{Ep}^{n+1} = U_p \Sigma_p V_p^T \text{ and } F_{Ep}^{n+1} = (F_{Ep}^{n+1})^{-1} F_p^{n+1} \]
Results
Discussion and Conclusion

- Limitations
  - A lot of our parameters were tuned by hand
  - This neglects interactions with the air
  - Using adaptive sparse grids would save memory

- Conclusion
  - Presenting a constitutive model and simulation technique
  - Demonstrating a wide variety of snow behaviors